Approximated Optimal Designs for a Simple Step-Stress Model with Type-II Censoring, and Weibull Distribution

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Abstract—This paper deals with theories for approximated optimal design for simple step-stress accelerated life testing (ALT) with type-II censoring, and Weibull distribution. Statistically approximated optimal ALT plans are developed to minimize the asymptotic variance of the maximum likelihood estimators (MLE) of the p-th percentile of lifetime at design stress. For the complex of calculating the expectation of order statistics, the Fisher information matrix for type-II censored data is approximated by that for type-I censored data. The approximated optimal plan doesn’t depend on the values of accelerating model parameters. Simulation results also show that the optimal stress levels are highest possible stress and lowest possible stress.

Keywords- accelerated life testing, step-stress, optimal design, type-II censoring, Cumulative exposure model

I. INTRODUCTION

Accelerated life testing is often used to obtain failure information quickly so that the life distribution of materials and products can be estimated. Specimens are tested at high stress levels to induce early failures, and then the failure information is related to that at design stress through a given stress-dependent distribution function. Such testing saves time and expense over testing at the design stress.

In most cases, additional savings may also be obtained from an optimal testing plan. As mentioned in [1, 2], most of ALT plans developed in the literature focus on tests with type-I censoring. This ALT plans rely on the planned value of model parameters, and they are not robust. However, it is common to conduct ALTs with type-II censoring. [3] obtained an optimum design for constant-stress ALTs with type-II censored data. Their optimum criterion is to minimize the variance of the best–linear-unbiased-estimation (BLUE) of the p-th percentile of lifetime at design conditions.

Here optimum designs are proposed for simple step-stress ALT with type-II censoring using Weibull distribution. The optimal criterion is to minimize asymptotic variance of the MLE of the p-th percentile of the lifetime distribution at design stress. For the complex of calculating expectation of order statistics, the Fisher information matrices are simplified by using large-sample properties of order statistics. Our results coincide with the results from [4]: the Fisher information matrices for type I and type II censored data from location-scale families are asymptotically equivalent. According to the expression of the approximated variance of the MLE of the p-th percentile of lifetime at design stress, the approximated optimal design doesn’t rely on the parameters of accelerating models. Simulation results also show that the optimal stress levels are highest possible stress and lowest possible stress. This property coincides with that of optimal designs for simple constant-stress ALT with type-II censoring using log-location-scale distribution.

The following part of this article is organized as follows. Section 2 describes models and assumptions. Section 3 presents an approximated optimal design. Section 4 summarizes the results.

II. MODELS AND ASSUMPTIONS

The models and assumptions are given as following.

1. At any stress level, the lifetimes \( T \) of test items are independent and follow a Weibull distribution with cumulative distribution function (CDF)
\[
Pr(T \leq t) = F(t; \mu, \sigma) = \left(1 - \exp\left(-\frac{\log(t) - \mu}{\sigma}\right)\right)
\]
where \( \mu \) is the location parameter of log, \( \sigma \) is the scale parameter of log(T), and \( F(t) \) is the CDF of a standardized minimum extreme value distribution, i.e. \( F(z) = 1 - \exp(-\exp(z)) \).

2. \( \mu \) is a linear function of (possibly transformed) stress. By standardizing the stress, the relationship becomes
\[
\mu = \mu(x) = \beta_0 + \beta_1 x
\]
where \( \beta_0 \) and \( \beta_1 \) are model parameters. Assume that \( x' \) is the original (possibly transformed) stress, let \( x = \frac{x' - x_d'}{x_h' - x_d'} \), where \( x_d' \) and \( x_h' \) are the design level, and the highest possible stress level,
respectively. Since \( x_{\min} \leq x \leq x_{\max}, 0 = x_{\min} \leq x_{\min} = 1 \).

3. A cumulative exposure model (CE), which relates the life distribution under step stress to that of constant stress, holds. That is, the remaining life of a test unit depends only on the cumulative exposure it has seen (see Nelson [5]).

4. \( n \) items are initially tested at stress \( x_i \) until \( r_i \) items fail, then the stress is increased to \( x_{i+1} \). The test continues until totally \( r \) items fail.

5. The goal of optimization is to determine \( r_i, x_i \) and \( x_{\max} \) when \( r \) is predetermined (\( 0 \leq x_{\min} < x_{\max} \leq 1 \)).

III. APPROXIMATED OPTIMAL DESIGN

A. Information Matrix

Since an extreme value distribution belongs to a location-scale distribution family. Firstly, we will derive the information matrix for a general log-location-scale distribution. Let \( \Phi(z) \) denote the CDF of a standard location-scale distribution, \( \phi(z) \) denote the corresponding probability density function, i.e. \( \phi(z) = \frac{d\Phi(z)}{dz}, \) and \( L(\beta_0, \beta, \sigma \mid D) \) denote the log likelihood function. Let

\[
\begin{align*}
   z_i &= \begin{cases} 
   \log(t_i - \mu(x_i)) & i \leq r_i \\
   \log(t_i - t_{i-s} - \mu(x_i)) & i > r_i 
   \end{cases} 
\end{align*}
\]

where \( t_i \) is the failure time of the \( i \)-th observation in the simple step-stress ALT, \( i = 1, \ldots, n \), \( s = t_i e^{\beta_0 + x_i} \). \( z_i \) is then the \( i \)-th order statistic from the distribution \( \phi(z) \) with the sample size of \( n \).

By some deduction, the second order partial derivatives of \( L(\beta_0, \beta, \sigma \mid D) \) with respect to \( \beta_0, \beta \), and \( \sigma \) are

\[
\begin{align*}
\sigma^2 \left[ \frac{\partial^2 L(\beta_0, \beta, \sigma \mid D)}{\partial \beta_0 \partial \beta} \right] &= \sum_{i=1}^{\min(r, n)} \left[ \frac{\phi(z_i) \phi(z_i) - (\phi(z_i))^2}{\phi(z_i)} \right] \\
&- (n-r) \frac{\phi(z_i)(1-\Phi(z_i)) + \phi(z_i)}{[1-\Phi(z_i)]^2} \\
\sigma^2 \left[ \frac{\partial^2 L(\beta_0, \beta, \sigma \mid D)}{\partial \beta \partial \sigma} \right] &= \sum_{i=1}^{\min(r, n)} \left[ \frac{\phi(z_i) \phi(z_i) - (\phi(z_i))^2}{\phi(z_i)} \right] \\
&- (n-r) \frac{\phi(z_i)(1-\Phi(z_i)) + \phi(z_i)}{[1-\Phi(z_i)]^2} \\
\sigma^2 \left[ \frac{\partial^2 L(\beta_0, \beta, \sigma \mid D)}{\partial \sigma^2} \right] &= \sum_{i=1}^{\min(r, n)} \left[ 2 \frac{\phi(z_i) \phi(z_i)(1-\Phi(z_i)) + \phi(z_i)}{[1-\Phi(z_i)]^2} \right] \\
&- (n-r) \frac{2 \phi(z_i)(1-\Phi(z_i)) + \phi(z_i)}{[1-\Phi(z_i)]^2} \\
\end{align*}
\]

The Fisher information matrix \( F^T \) of \( \beta_0, \beta \), and \( \sigma \) then satisfies

\[
\frac{F^T}{n} = \frac{\sigma^2}{n} \begin{bmatrix}
   \mathbf{E} \left( \frac{\partial^2 L}{\partial \beta_0 \partial \beta} \right) & \mathbf{E} \left( \frac{\partial^2 L}{\partial \beta_0 \partial \sigma} \right) & \mathbf{E} \left( \frac{\partial^2 L}{\partial \beta \partial \sigma} \right) \\
   \mathbf{E} \left( \frac{\partial^2 L}{\partial \beta \partial \sigma} \right) & \mathbf{E} \left( \frac{\partial^2 L}{\partial \sigma^2} \right) & \mathbf{E} \left( \frac{\partial^2 L}{\partial \beta \partial \sigma} \right) \\
   \mathbf{E} \left( \frac{\partial^2 L}{\partial \beta \partial \sigma} \right) & \mathbf{E} \left( \frac{\partial^2 L}{\partial \beta \partial \sigma} \right) & \mathbf{E} \left( \frac{\partial^2 L}{\partial \sigma^2} \right)
\end{bmatrix}
\]

where \( \sigma^2 \) denotes \( L(\beta_0, \beta, \sigma \mid D) \).
The direct calculation of $F^f$ is complex because of the order statistics. So, we will approximate $F^f$ to a simple form by using large sample properties of the order statistics.

**Theorem 1.** $Y, Y_1, \cdots, Y_n$ are i.i.d. from a distribution with CDF of $\Phi(.)$. $Y^{(i)}$ is the $i$-th order statistic, $i=1,\cdots,n$. $g(.)$ is a continuous function, and $E(|g(Y)|) < \infty$. Let $\xi=\Phi^{-1}(1-\pi_c)$. If \( r_i \rightarrow \pi_i, \) and \( r_i \rightarrow 1-\pi_c \) as \( n \rightarrow \infty \), then

$$
\eta=\Phi^{-1}(1-\pi_c). \quad \text{If} \quad \frac{r_i}{n} \rightarrow \pi_i, \quad \frac{r}{n} \rightarrow 1-\pi_c \quad \text{as} \quad n \rightarrow \infty, \quad \text{then}
$$

\[
\left(1\right) \quad \frac{1}{n} \sum_{i=1}^{n} E[g(Y^{(i)})] \rightarrow E\{g(Y) \cdot I(Y \leq \xi)\}
\]

\[
\left(2\right) \quad \frac{1}{n} \sum_{i=1}^{n} E[g(Y^{(i)})] \rightarrow E\{g(Y) \cdot I(\xi < Y \leq \eta)\}
\]

\[
\left(3\right) \quad \frac{n-r}{n} E[g(Y^{(i)})] \rightarrow (1-\pi_c) g(\eta)
\]

\[
\left(4\right) \quad E[g(Y^{(i)})] \rightarrow g(\xi)
\]

as \( n \rightarrow \infty \).

**Proof:** We only prove (1), (2)-(3) can be proved similarly.

Since \( \frac{r_i}{n} \rightarrow \pi_i \), as \( n \rightarrow \infty \), it is obvious that \( Y^{(i)} \rightarrow \xi \) as \( n \rightarrow \infty \).

\[
\frac{1}{n} \sum_{i=1}^{n} E[g(Y^{(i)})] = \frac{1}{n} \sum_{i=1}^{n} E[g(Y^{(i)}) | Y^{(i)} \leq Y^{(i)}] = E\{\frac{1}{n} \sum_{i=1}^{n} E[g(Y^{(i)}) | Y^{(i)} \leq Y^{(i)}]\} = E\{E[g(Y^{(i)}) | Y^{(i)} \leq Y^{(i)}]\} = E[g(Y) I(Y \leq \xi)]
\]

as \( n \rightarrow \infty \).

According to the above theorem,

\[
\frac{\sigma^2}{n} F^f \approx \frac{\sigma^2}{n} F^f \quad (n \rightarrow \infty, \frac{r_i}{n} \rightarrow \pi_i, \frac{r}{n} \rightarrow 1-\pi_c)
\]

where $F^f$ is the Fisher information matrix based on a step-stress ALT with type-I censoring, i.e. $n$ items are initially tested at $x_i$ until time $\exp(\sigma \xi + \mu(x_i))$, then the stress is increased to $x_{i+1}$, the test continues until time $\exp(\sigma \xi + \mu(x_{i+1})) + \exp(\sigma \xi + \mu(x_{i})) - \exp(\sigma \xi + \mu(x_{i}))$. Here $\xi=\Phi^{-1}(1-\pi_c)$, $\eta=\Phi^{-1}(1-\pi_c)$, and $\pi_i = r_i / n$, $1-\pi_c = r / n$.

Next we will give the expression of $F^f$ when $\Phi(.)$ is $EV(.)$. Let

\[
\frac{\sigma^2}{n} F^f = \begin{bmatrix}
A_{11} & A_{12} & A_{13} \\
A_{21} & A_{22} & A_{23} \\
A_{31} & A_{32} & A_{33}
\end{bmatrix}
\]

by some calculation, we have

\[
A_{11} = 1 - \pi_c
\]

\[
A_{12} = \int_0^{-x_c} \ln(-\ln(\pi)) \ln(1-t) \ dt - \pi_c \ln(-\ln(\pi)) \ln(1-t)
\]

\[
A_{13} = x_c (\pi_1 + (1-\pi_1) \ln(1-\pi_1)) + x_c (1-\pi_1 - (1-\pi_1) \ln(1-\pi_1))
- (x_c - x_c) (\ln(\pi_1))^a (\ln(\pi_1))^\sigma
- (x_c - x_c) (\ln(\pi_1))^\sigma \int_{x_c}^{1-x_c} \ln(-\ln(1-t))^\sigma \ dt
\]

\[
A_{22} = \int_0^{-x_c} \ln(-\ln(1-t)) \ln(1-t) \ dt + \int_{-x_c}^{1} \ln(-\ln(1-t)) \ln(1-t) \ dt
- 2 \int_0^{-x_c} \ln(-\ln(1-t)) \ ln(1-t) \ dt - \pi_c \ln(-\ln(1-t))\ln(1-t)
- \pi_c \ln(-\ln(1-t))\ln(1-t) \ 
\]

\[
A_{33} = (x_c - x_c) \ln(1-\pi_1) - (1-\pi_1)
+ x_c \int_0^{x_c} \ln(-\ln(1-t)) \ln(1-t) \ dt
+ x_c \int_{-x_c}^{1-x_c} \ln(-\ln(1-t)) \ln(1-t) \ dt
- x_c \pi_1 \ln(1-\pi_1) \ln(-\ln(1-t))
- (x_c - x_c) \ln(1-\pi_1)^\sigma \int_{x_c}^{1-x_c} \ln(-\ln(1-t))\ln(1-t) \ dt
+ \pi_c \ln(-\ln(1-t))\ln(1-t)
\]

From the above expressions, $\frac{\sigma^2}{n} F^f$ does not contain accelerating model parameters, $\beta_1, \beta_2$. It implies that the approximated optimum design doesn’t rely on $\beta_1$ and $\beta_2$, either.

**B. Objective Functions**
Let $ApVar(\ln \hat{\lambda}_p)$ denote the approximated variance of the MLE of the logarithm $p$-th percentile of lifetime at design stress, we have
\[
\ln \hat{\lambda}_p = \beta_0 + (\ln(-\ln(1-p)))\sigma
\]  
(4)

$ApVar(\ln \hat{\lambda}_p) = [1,(\ln(-\ln(1-p)))0(F')]^{-1}[1,(\ln(-\ln(1-p)))0]^T$

By some deduction, we find that $ApVar(\ln \hat{\lambda}_p)$ only relates on $x_i/x_2$ and $\pi_1$. Let $\zeta = x_i/x_2$, then
\[
ApVar(\ln \hat{\lambda}_p) = \frac{\sigma^2}{n} v(\zeta, \pi_1)
\]

with
\[
v(\zeta, \pi_1) = \frac{M g_p(\zeta, \pi_1) - (g_p(\zeta, \pi_1)\ln(-\ln(1-p)) - g_p(\zeta, \pi_1))^2}{M g_p(\zeta, \pi_1) - (A_1 g_p(\zeta, \pi_1) + A_2 g_p(\zeta, \pi_1) - 2 A_2 g_p(\zeta, \pi_1) g_p(\zeta, \pi_1))}
\]

where
\[
M_1 = A_2 + (\ln(-\ln(1-p)))^2 A_1 - 2 (\ln(-\ln(1-p))) A_2,
\]
\[
M_2 = A_2 A_1 - A_2^2,
\]
\[
g_p(\zeta, \pi_1) = \frac{A_1}{x_i}, i = 1, 2, \quad g_p(\zeta, \pi_1) = \frac{A_1}{x_i^2}
\]

Thus, the optimal problem is
\[
\text{Min} \quad v(\zeta, \pi_1)
\]
\[
st. \quad 0 \leq \zeta \leq 1
\]
\[
0 \leq \pi_1 \leq 1 - \pi_c
\]

C. Approximated Optimal Design

According to equation (5), approximated optimal designs for estimating $\hat{\lambda}_{0.1}$, $\hat{\lambda}_{1}$, and $\hat{\lambda}_{2}$ are shown in Fig. 2- Fig. 4 with $\pi_c = 0.2$, $0.5$, $x_i = 0.4$, $0.6$, $0.8$, and $\sigma = 0.5$.

However, if work stress level is included in planning ALT with type-II censoring, it is naturally to test all items at the work stress. Also, when consider the optimal problem for different $\pi_c$ and $\sigma$, we find that $v(\zeta, \pi_1)$ is increasing with respect to $\zeta$ when $\zeta \in (0,1)$, and convex with respect to $\pi_1$ when $\pi_1 \in (0,1 - \pi_c)$. This phenomena is shown in Fig. 1. So, we need to predetermine the lowest possible stress level $x_i$ ($x_i \in (0,1)$). The choice of $x_i$ relies on time consuming consideration or other possible considerations. The optimal problem then becomes

\[
\text{Min} \quad v(\zeta, \pi_1)
\]
\[
st. \quad x_i \leq \zeta \leq 1
\]
\[
0 \leq \pi_1 \leq 1 - \pi_c
\]

(5)\n
Actually, according to expression (4), $\hat{\theta}_p$ relies more on $\sigma$ than on $\beta_0$ when $p$ is extremely large or small. So,
approximated optimal design is robust to variation in $\sigma$ only when $p$ is moderate.

**Example:** $n = 30, r = 24, \sigma = 1$, and $x_i = 0.6$, the estimating objective is $\hat{I}_{0.01}$. Since $\hat{\pi}_i = 0.2$, then the optimal $\pi_i = 0.43$. The approximated design is: 30 items are initially tested at $x = 0.6$ until 30 × 0.43 = 13 items fail, then the stress is improved to $x = 1$, testing continues until other 9 items fail.

IV. CONCLUSIONS

In this article, we have developed an approximated optimum design for simple step-stress ALT with type-II censoring using Weibull distribution. In this design, we need to predetermine a lowest possible stress level. The optimum stress levels for testing are the lowest possible stress level and the highest possible stress level. This design doesn't depend on the value of model parameters $\beta_0$ and $\beta_1$. It is also robust to variation in $\sigma$ when $p$ is moderate.

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