Optimum Design for Type-I Step-stress Accelerated Life Tests of Two-parameter Weibull Distributions

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Abstract—In this paper, we focus on the two-parameter Weibull distribution with censored sample based on TFR model under simple step-stress accelerated life testing. We create the optimum design for TFR model using the likelihood function and Fisher information matrix. Finally, we illustrate the feasibility of this method through an example.

Keywords- Weibull distribution; Type-I censored and simple step-stress accelerated life testing; TFR model; optimum design

I. INTRODUCTION

With regard to the optimum design, there is little discussion about the optimum design for TFR model. Reference [1] gave the optimum design with total failure sample from Weibull distribution for step-stress accelerated life test. This paper focuses on the two-parameter Weibull distribution with censored sample under simple step-stress accelerated life test (SSALT). It presents the optimum design for TFR model. Section 2 gives some basic assumptions and related conclusions. Section 3 gives the optimum design for Type-I SSALT of Weibull distribution. Section 4 gives an example to show the effectiveness of the method.

II. BASIC ASSUMPTIONS AND RELATED CONCLUSIONS

Assumption 1. Life distribution: The life-time under constant stress $S$ follows the Weibull distribution $Wei(\theta, \beta)$ with cumulative distribution function(CDF)

$$F(t) = 1 - \exp\{-\left(\frac{t}{\theta}\right)^\beta\},$$

where $\theta$ and $\beta$ are the scale and shape parameters respectively.

Assumption 2. Accelerated tampered model: The hazard rate function (HRF) for the two-step SSALT is assumed to be

$$\lambda^*(t) = \begin{cases} 
\lambda(t), & 0 \leq t \leq \tau_1 \\
\alpha \lambda(t), & t > \tau_1 
\end{cases},$$

where $\lambda(t)$ is the HRF under $S_1$, $\tau_1$ is the time transition point.

Assumption 3. Accelerated function: The characteristic life and the stress have the following logarithmically linear relation:

$$\ln \theta = a + b \varphi(S),$$

where $\varphi$ is a known function, and $\ln \theta = \varphi(S)$.

Hence we have the following conclusions which are needed in our discussion.

Proposition 1. Under assumptions 1 and 2, if the lifetime under $S_1$ follows $Wei(\theta_1, \beta)$, then density function for the two-step SSALT is

$$f^*(y) = \begin{cases} 
\frac{\beta}{\theta_1^\beta} y^{\beta-1} \exp\{-\left(\frac{y}{\theta_1}\right)^\beta\}, & y \leq \tau_1 \\
\frac{\beta}{\theta_2^\beta} y^{\beta-1} \exp\{-\left(\frac{y}{\theta_2}\right)^\beta - (1 - \alpha)(\frac{\tau_1}{\theta_2})^\beta\}, & y > \tau_1
\end{cases}.$$  

Proposition 2. Under assumptions 1 and 2, we have

$$\beta_2 = \beta_1 \cdot \frac{\ln \theta_1}{\ln \theta_2} $$

Proposition 3. Under assumptions 1-3, $\alpha$ satisfies the log-linear relationship

$$\alpha = \exp\{b\beta[\varphi_1 - \varphi_2]\}.$$  

III. OPTIMUM TWO-STEP SSALT AND TYPE I CENSORING

Suppose $n$ products are initially placed on test at $S_1$. Assume that $r_1$ products fail under the stress $S_1$ and the ordered failure times are $0 < t_1 \leq t_2 \leq \cdots \leq t_{r_1} \leq \tau_1$. $r_2$ products fail under the stress $S_2$ and the ordered failure times are $\tau_1 \leq t_{r_1+1} \leq t_{r_1+2} \leq \cdots \leq t_{r_1+r_2} \leq \tau$.  

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We introduce the following notations:

\[ \Gamma(x|\gamma) = \int_x^\infty \frac{1}{\Gamma(\gamma)} y^{\gamma-1} \exp(-y) \, dy, \]

\[ g_1(a, b, \beta|m) = \exp\left[-(1 - \alpha)\left(\frac{m}{\theta_1}\right)^\beta\right] \left\{ \Gamma\left(\frac{m}{\theta_2^\beta}\right) - \Gamma\left(\frac{m}{\theta_1^\beta}\right) \right\}, \]

\[ g_2(a, b, \beta) = \exp\left[-(1 - \alpha)\left(\frac{\theta_1}{\theta_2}\right)^\beta\right] \left\{ \exp\left(\frac{\theta_1}{\theta_2^\beta}\right) - \exp\left(-\frac{\theta_1}{\theta_2^\beta}\right) \right\}, \]

where \( \alpha \) is given in (5). Then the asymptotic variance of the maximum likelihood estimate of the \( \theta \) quartile life time under the constant stress \( \varphi_0 \) can be expressed as

\[ \text{Avar}(\ln \hat{\theta}_p) = \left(1 \varphi_0 - \frac{1}{\beta^2} \Phi^{-1}(p)\right) \sum \left( \varphi_0 - \frac{1}{\beta^2} \Phi^{-1}(p) \right) \]

where \( \Phi(x) \) is the CDF of standard extreme value distribution and \( \Sigma = F^{-1} \) where \( F \) is the Fisher information matrix of the likelihood estimation of \( \alpha, \beta \) and \( \beta \), that is,

\[ F = \left( \begin{array}{ccc} \frac{\partial^2 \ln L}{\partial \alpha^2} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} & \frac{\partial^2 \ln L}{\partial \alpha \partial \beta} \\ \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta^2} & \frac{\partial^2 \ln L}{\partial \beta^2} \\ \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta \partial \alpha} & \frac{\partial^2 \ln L}{\partial \beta^2} \end{array} \right) = \left( \begin{array}{ccc} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{array} \right) \]

and

\[ A_{11} = \beta^2 \Gamma\left(\frac{\beta}{\theta_1}\right)^2 + \beta^2 g_1(a, b, \beta|2) + \beta \tau (1 - \alpha) f^*(\tau_1) + \beta \tau f^*(\tau), \]

\[ A_{12} = \beta^2 \phi_0 \Gamma\left(\frac{\beta}{\theta_1}\right)^2 + \beta^2 \phi_0 g_1(a, b, \beta|2) + \beta \tau (1 - \alpha) f^*(\tau_1) + \beta \tau f^*(\tau), \]

\[ A_{13} = F_1(\gamma_1) - (1 - \beta \ln \theta_1) \Gamma\left(\frac{\beta}{\theta_1}\right)^2 - \beta \Gamma\left(\frac{\beta}{\theta_1}\right)^2 - \beta \Gamma\left(\frac{\beta}{\theta_2}\right)^2 - \beta \Gamma\left(\frac{\beta}{\theta_3}\right)^2 - \beta \Gamma\left(\frac{\beta}{\theta_4}\right)^2 - \beta \Gamma(2 - \frac{1}{\beta}) - \beta \Gamma\left(\frac{\beta}{\theta_1}\right)^2 - \beta \Gamma(2 - \frac{1}{\beta}) - \beta \Gamma\left(\frac{\beta}{\theta_2}\right)^2 - \beta \Gamma(2 - \frac{1}{\beta}) - \beta \Gamma\left(\frac{\beta}{\theta_3}\right)^2 - \beta \Gamma(2 - \frac{1}{\beta}) - \beta \Gamma\left(\frac{\beta}{\theta_4}\right)^2 - \beta \Gamma(2 - \frac{1}{\beta}) - \beta \Gamma(2 - \frac{1}{\beta}) - \beta \Gamma(2 - \frac{1}{\beta}) - \beta \Gamma(2 - \frac{1}{\beta}) \]

\[ A_{22} = \beta^2 \phi_0 \Gamma\left(\frac{\beta}{\theta_1}\right)^2 + \beta^2 \phi_0 g_1(a, b, \beta|2) + \beta \tau (1 - \alpha) f^*(\tau_1) + \beta \tau f^*(\tau), \]

\[ A_{23} = \phi_0 F_1(\gamma_1) - \phi_0 (1 - \beta \ln \theta_1) \Gamma\left(\frac{\beta}{\theta_1}\right)^2 - \phi_0 \Gamma\left(\frac{\beta}{\theta_1}\right)^2 - \phi_0 \Gamma\left(\frac{\beta}{\theta_2}\right)^2 - \phi_0 \Gamma(2 - \frac{1}{\beta}) - \phi_0 \Gamma\left(\frac{\beta}{\theta_1}\right)^2 - \phi_0 \Gamma(2 - \frac{1}{\beta}) - \phi_0 \Gamma\left(\frac{\beta}{\theta_2}\right)^2 - \phi_0 \Gamma(2 - \frac{1}{\beta}) - \phi_0 \Gamma\left(\frac{\beta}{\theta_3}\right)^2 - \phi_0 \Gamma(2 - \frac{1}{\beta}) - \phi_0 \Gamma\left(\frac{\beta}{\theta_4}\right)^2 - \phi_0 \Gamma(2 - \frac{1}{\beta}) - \phi_0 \Gamma(2 - \frac{1}{\beta}) - \phi_0 \Gamma(2 - \frac{1}{\beta}) - \phi_0 \Gamma(2 - \frac{1}{\beta}) \]

\[ A_{32} = \beta^2 \phi_1 \Gamma\left(\frac{\beta}{\theta_1}\right)^2 + \beta^2 \phi_1 g_1(a, b, \beta|2) + \beta \tau (1 - \alpha) f^*(\tau_1) + \beta \tau f^*(\tau), \]

\[ A_{33} = \phi_1 F_1(\gamma_1) - \phi_1 (1 - \beta \ln \theta_1) \Gamma\left(\frac{\beta}{\theta_1}\right)^2 - \phi_1 \Gamma\left(\frac{\beta}{\theta_1}\right)^2 - \phi_1 \Gamma\left(\frac{\beta}{\theta_2}\right)^2 - \phi_1 \Gamma(2 - \frac{1}{\beta}) - \phi_1 \Gamma\left(\frac{\beta}{\theta_1}\right)^2 - \phi_1 \Gamma(2 - \frac{1}{\beta}) - \phi_1 \Gamma\left(\frac{\beta}{\theta_2}\right)^2 - \phi_1 \Gamma(2 - \frac{1}{\beta}) - \phi_1 \Gamma\left(\frac{\beta}{\theta_3}\right)^2 - \phi_1 \Gamma(2 - \frac{1}{\beta}) - \phi_1 \Gamma\left(\frac{\beta}{\theta_4}\right)^2 - \phi_1 \Gamma(2 - \frac{1}{\beta}) - \phi_1 \Gamma(2 - \frac{1}{\beta}) - \phi_1 \Gamma(2 - \frac{1}{\beta}) - \phi_1 \Gamma(2 - \frac{1}{\beta}) \]

Particularly, let \( \beta = 1 \), then the Weibull distribution is the exponential distribution. Therefore, given the use-condition stress \( S_0 \), the two accelerated stress levels, \( S_1 \) and \( S_2 \), the sample size \( n \) and the censoring time \( T \), we can get from the expression (6) that the optimum transition time \( T_1 \) satisfies

\[ \left( \frac{\varphi_2 - \varphi_0}{\varphi_0 - \varphi_1} \right)^2 = \left( \frac{n_2}{n_1} \right)^2 \left[ \frac{p_1}{p_2} \right]^2 \left[ \frac{p_1}{p_2 - (1 - p_1)} \right]^2, \]

where

\[ p_1 = 1 - \exp\left(-\frac{\tau_1}{\theta_1}\right), \]

\[ p_2 = 1 - \exp\left(-\frac{T - \tau_1}{\theta_1}\right) \]

and the solution of (8) is unique. The result is consistent with that of optimum design for Type-I SSALT of exponential distribution under CE model in reference [2].

IV. AN EXAMPLE

Let the use-condition stress \( S_0 = 400K \), the acceleration temperature levels are 500K and 800K, the sample size is 1000, and the censoring time is 3000h. According to experience (or historical data), let \( \hat{\alpha} = 10, \hat{\beta} = -1, \hat{\beta} = 2 \). Then from (6) we can get the optimum design, i.e. the transition point \( T_1 = 1221.2h \). At this time, \( \text{Avar}(\ln \hat{\theta}_p) = 0.9458 \). The following graph shows the relationship between \( \text{Avar}(\ln \hat{\theta}_p) \) and \( \tau_1 \).
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