Abstract—This paper proposes to design an $\bar{X}$ control chart through a data envelopment analysis (DEA) based multicriteria branch and bound algorithm. The proposed multicriteria model avoids the difficulty of estimating costs associated with false alarm and operation under out-of-control status, and reduces the need for the decision maker to specify the statistical bounds on two types of errors and to provide preference information during the control chart design process. Through appropriate branching and bounding procedures, the DEA-based multicriteria branch and bound algorithm can quickly target the solution region where Pareto optimal solutions exist. Instead of involving the decision makers during the whole design process, few Pareto optimal solutions with useful tradeoff information can be presented to decision makers.

Keywords—Multicriteria decision analysis, branch and bound, data envelopment analysis

I. INTRODUCTION

Control chart design has been extensively researched in the literature since the first economic design model proposed by Duncan (1956). A pure economic design of the control chart based on total cost could be unrealistic or inappropriate for decision making due to the unsatisfactory statistical properties. These weaknesses were pointed out by Woodall (1986). Later, Saniga (1989) proposed a statistical-economic model of designing control chart by imposing four statistical constraints in the optimization model. These constraints are, an upper bound on type I error, an upper bound on the average time to signal (ATS) an out of control situation, a lower bound on the power probability, and a lower bound on the power for some customer specified shift sizes.

In addition to the weaknesses suggested by Woodall, criticism for the economic design of control chart has also come from the difficulty of some estimation costs in the original Duncan's model. To deal with this issue, Castillo et al. (1996) proposed a multicriteria optimal design of $\bar{X}$ control chart through an interactive multicriteria decision making (MCDM) algorithm by using a linear value function to approximate the customers’ decision preference. The model involves three criteria (i.e., the expected number of false alarms, the average time to signal, and the sampling cost per cycle) and two statistical constraints (an upper bound on false alarm and a lower bound on power). Such formulation avoids the difficulties of estimating the costs associated with false alarms, operation in out-of-control status, and investigation of assignable causes. However, the linear approximation of the value function may not represent the utility function of the decision, and the decision maker may have little knowledge to specify the statistical bounds in the model and to provide tradeoff/preference information between different designs.

This paper investigates the design of $\bar{X}$ control chart under a multicriteria branch and bound framework from the perspective of evaluating the multiple criteria in terms of relative efficiency used in data envelopment analysis (DEA). Similar to Castillo et al. model, only measurable costs are utilized in the model, and other immeasurable costs and long-term quality performance indices are represented by higher average run length $ARL_0$ and the detection power $p$. Moreover, by using Pareto optimality or non-dominance concepts, there is no need for the decision makers to specify the statistical bounds. Multiple Pareto optimal solutions will be presented to the decision makers, and those designs which are not satisfactory in terms of the bounds could be easily pruned. The decision variables in the control chart design are the sample size $n$, control limits $k$ in terms of a known process standard deviation $\sigma$, and the sampling frequency in terms of the interval $h$ of two successive samples. One possible design of the control chart is a combination of $n$, $h$, and $k$, which is denoted by $s = (n, h, k)$.

II. PROBLEM FORMULATION

Based on the assumptions provided by Duncan (1956), some quantitative performance can be easily derived. (1) the type I error or the probability of false alarm $\alpha$ for each sample is,

$$\alpha(s) = \frac{1}{ARL_0} = 2\int_{-\infty}^{z_x} \phi(z)dz,$$  \hspace{1cm} (1)

where $\phi(z)$ is the standard normal density function; (2) the detection power is $p$ for each sample after the process is in out-of-control status, which is,

$$p(s) = \int_{-\infty}^{z^*} \phi(z)dz + \int_{-\infty}^{z^*} \phi(z)dz,$$  \hspace{1cm} (2)
where $\delta$ is the mean shift in terms of process standard deviation. In the traditional economic control design, all performance criteria are converted to a single objective to be minimized. However, it is very difficult to estimate the cost of false alarm ($\alpha$), and the cost of releasing defective products to customer (type II error or $1-p$), which could be unpredictable and immeasurable, we propose to simultaneously maximize the average run length $ARL_s$ (equal to $1/\alpha$), maximize the detection power, and minimize only the sampling related costs. Such multicriteria formulation will present the decision makers many Pareto optimal solutions, with the tradeoff information among the three objectives. Since many alternative designs will be available to the decision makers, it is not necessary to specify an upper bound for $\alpha$ and a lower bound for $p$, which could also be difficult for some decision makers who have little understanding of these specialized concepts.

The mathematical formulation of the multicriteria $\bar{X}$ control chart is as,

$$\begin{align*}
\text{Max} & \quad ARL_s(s) \\
\text{Max} & \quad p(s) \\
\text{Min} & \quad C(s)
\end{align*}$$

(3)

where $s = (n, h, k)$ which are the decision variables, and $C(s) = (a_i + a_n)/h$, $a_i$ and $a_n$ are respectively the fixed cost and varying cost of sampling.

III. METHODOLOGY

The multicriteria control chart design problem is solved using a DEA-based multicriteria branch and bound algorithm. In traditional single objective branch and bound algorithm, subproblems are pruned when the objective value is worse than the achieved bounds or by infeasibility. For MCDM, the bound for one single objective is not enough to prune some solution regions. Instead the Pareto frontier needs to be identified such that some solution regions without any Pareto solutions could be pruned. The DEA method (Cook and Seiford, 2009) is applied to obtain and prune the Pareto optimal solutions in the proposed multicriteria branch and bound algorithm.

The DEA-based multicriteria branch and bound algorithm proceeds as follows:

**Step 1** Branching procedure

Divide the feasible solution region into appropriate subregions;

**Step 2** Bounding procedure

For each subregion, obtain the Pareto optimal solutions using DEA method.

- If no Pareto optimal solutions exist in one subregion, prune this region by infeasibility;

- If Pareto optimal solutions exist, these solutions form the set of bounds.

**Step 3** Combine all the Pareto optimal solutions, and those which are dominated by other Pareto solutions are pruned. The remaining Pareto optimal solutions are stored as the updated set of bounds.

**Step 4** Retrieve the refined solution regions from the updated Pareto frontier.

**Step 5** Terminate if the stopping criterion is satisfied. Otherwise, go to step 1.

In the multicriteria control chart design, an initial feasible solution region based on engineering experience or experts’ opinion could be determined. This feasible solution region is divided into a few subregions and the relative efficiency of some representative solutions is evaluated using DEA method. More specifically, the minimization objectives (cost type) are treated as inputs and maximization objectives (benefit type) are treated as outputs of a decision making unit in DEA. The solutions with relative efficiency equal to one form the Pareto optimal frontier, and those regions where the solutions’ relative efficiency lower than one can be pruned. After some iterations, a refined region where Pareto optimality exists becomes the target solution region and DEA or some other multicriteria optimization methods could be applied to achieve the final Pareto optimal solutions, and only few workable Pareto optimal solutions would be presented to the decision makers.

IV. NUMERICAL EXAMPLE

The formulation of (3) in section 2 is solved by applying the DEA-based multicriteria branch and bound algorithm. From practical engineering experience, the solution regions are confined in a limited three dimensional space. The control limit $k$ in terms of the process standard deviation takes value from 2.5 to 3.5 with a step size of 0.2. The sampling interval $h$ takes value from 0.3 to 0.5 with step size of 0.1, and the sample size $n$ takes values from 20 to 30. Since the initial iterations are to detect the solution regions where Pareto optimal solutions locate, the step size could be relatively large. The more precise branching processes could be implemented to the refined solution regions.

After applying the DEA to the well-dispersed solutions in the two branched subregions respectively, a total number of 30 Pareto optimal solutions with 15 in each subregion (represented by the star sign in Figure 1) are obtained. These Pareto optimal solutions are located around the region where the sampling interval value is 0.5. The Pareto optimal solutions and the objective values in objective space are plotted in Figure 1 and Figure 2.

These 30 Pareto optimal solutions are obtained separately in the two subregions, and five of them are dominated by other Pareto optimal solutions when they are evaluated simultaneously using DEA method. An updated Pareto optimal frontier with 25 Pareto optimal solutions is achieved in the first iteration of the algorithm.
Based on the information from iteration one, a refined solution region is identified with a total of 150 solutions which are located around the Pareto solutions obtained in iteration one (with smaller step size when dividing this refined solution region). In the bounding process, DEA is applied and 11 Pareto optimal solutions are obtained for each subregion. The Pareto optimal solutions and the objective values in objective space are plotted in Figure 3 and Figure 4.

The updated Pareto frontier with 21 solutions is achieved after evaluating all the 22 Pareto optimal solutions simultaneously. After two iterations, it seems we have identified the region where Pareto optimality exists. If this updated Pareto frontier is satisfactory to the decision makers, the algorithm terminates. Otherwise, a more precise searching process within the refined solution region could be determined and the branching and bounding procedures can be repeated until the stopping criterion is satisfied.

V. DISCUSSION AND CONCLUSION

The control chart design is formulated as a MCDM problem without specifying the statistical bounds and estimating immeasurable cost associated with false alarm and operation under out-of-control status. The MCDM problem is solved using the proposed DEA-based multicriteria branching and bound algorithm and a reduced Pareto optimal solution set is presented to decision makers for choosing a favorable solution based on their preferences.

The application of DEA can quickly locate the solution region where Pareto optimality exists even though it could miss some convex dominated Pareto optimal solutions (Stewart, 1996). The refined solution regions can be further decomposed iteratively until a satisfying Pareto optimal solution set is achieved or any other stopping conditions are met. In the proposed algorithm, only a small portion of solutions in the whole feasible solution region needs to be evaluated in order to know the target Pareto solution region, and Pareto optimal solution searching efforts could be focused more on the refined solution regions. When the refined solution regions are identified, more delicate branching procedures could be carried out or even other multicriteria selection optimization methods could be applied to this refined solution region. The algorithm provides a good alternative to solve multicriteria control chart design problem in an efficient and practical way.

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REFERENCES


